Ex. Find the sol^n of D.E.

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec} y/x = 0 \quad ; \; y = 0 \text{ when } x = 1$$

Sol^n:- The given D.E. is:

$$\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\frac{y}{x} \qquad \text{—— (A)}$$

which is the form of $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ is homogeneous

put $y = vx$ —— (1)

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{— (2)}$$

$$(A) \Rightarrow v + x\frac{dv}{dx} = \frac{vx}{x} - \operatorname{cosec}\frac{vx}{x} = v - \operatorname{cosec} v$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = \cancel{v} - \operatorname{cosec} v$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1}{\sin v} \Rightarrow -\sin v\, dv = \frac{dx}{x}$$

$$\Rightarrow -\int \sin v\, dv = \int \frac{dx}{x} + c$$

$$\Rightarrow \cos v = \log x + c \Rightarrow \cos\frac{y}{x} = \log x + c \qquad \text{—— (3)}$$

put $y(1) = 0$ in (3)

$$\cos\left(\frac{0}{1}\right) = \log(1) + c \Rightarrow \boxed{c = 1}$$

$$\cos\left(\frac{y}{x}\right) = \log x + 1$$

which is the required sol^n.

EQUATION REDUCIBLE TO HOMOGENEOUS

Differential eqn of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad \text{---} (*)$$

are not homogeneous but can be reduced to homogeneous by a suitable transformation.

When $\frac{a}{a'} \neq \frac{b}{b'}$

Put $x = x'+h$, $y = y'+k$

; h & k are constants

Then $dx = dx'$ & $dy = dy'$

$(*) \Rightarrow$

$$\frac{dy'}{dx'} = \frac{a(x'+h)+b(y'+k)+c}{a'(x'+h)+b'(y'+k)+c'}$$

$$= \frac{ax'+by'+(ah+bk+c)}{a'x'+b'y'+(a'h+b'k+c')}$$

Put $ah+bk+c=0$

$a'h+b'k+c'=0$

& solving these

$\left[\frac{a}{a'} \neq \frac{b}{b'}\right]$

So that

$$\frac{dy'}{dx'} = \frac{ax'+by'}{a'x'+b'y'}$$ which homo.

in x' & y' & can be solved

Put $x' = x-h$, $y' = y-k$.

When $\frac{a}{a'} = \frac{b}{b'} = k$.

ie $a = a'k$ & $b = b'k$.

$$(*) \Rightarrow \frac{dy}{dx} = \frac{k(a'x+b'y)+c}{a'x+b'y+c'}$$

Put $a'x+b'y = t$

$a' + b'\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow b'\frac{dy}{dx} = \frac{dt}{dx} - a'$

So that $\Rightarrow \frac{dy}{dx} = \frac{1}{b'}\left[\frac{dt}{dx} - a'\right]$

$$\Rightarrow \frac{1}{b'}\left[\frac{dt}{dx} - a'\right] = \frac{kt+c}{t+c'}$$

$$\Rightarrow \frac{dt}{dx} - a' = \frac{b'(kt+c)}{t+c'}$$

$$\Rightarrow \frac{dt}{dx} = \frac{b'(kt+c)}{t+c'} + a'$$

Ex.1 Solve the differential eq^n

$$(3x-7y-3)dy + (7x-3y-7)dx = 0$$

Sol^n:- The given differential eq^n is

$$\frac{dy}{dx} = \frac{-(7x-3y-7)}{3x-7y-3} \quad \text{——} \quad (1)$$

comparing (1) with $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$

$$\therefore \frac{a}{a'} = \frac{-7}{3}, \quad \frac{b}{b'} = \frac{3}{-7} \Rightarrow \frac{a}{a'} \neq \frac{b}{b'}$$

put $x = x'+h$ & $y = y'+k$ in (1)

$dx = dx'$ & $dy = dy'$

$$(1) \Rightarrow \frac{dy'}{dx'} = -\frac{7(x'+h)-3(y'+k)-7}{3(x'+h)-7(y'+k)-3} = -\frac{7x'-3y'+(7h-3k-7)}{3x'-7y'+(3h-7k-3)} \quad \text{——} \quad (2)$$

put $7h-3k-7=0$

$3h-7k-3=0$

solving these, we get:

$$\frac{h}{9-49} = \frac{k}{-21+21} = \frac{1}{-49+9}$$

$$\Rightarrow \frac{h}{-40} = \frac{k}{0} = \frac{1}{-40}$$

$$\Rightarrow \boxed{h=1,\ k=0}$$

$\because ah+bk+c=0$

$a'h+b'k+c'=0$

$$\frac{h}{bc'-b'c} = \frac{k}{ca'-c'a} = \frac{1}{ab'-a'b}$$

So eq^n (2) $\Rightarrow$

$$\frac{dy'}{dx'} = -\frac{7x'-3y'}{3x'-7y'} \quad \text{——} \quad (3)$$

eq^n (3) is homogeneous of zero degree

put $y' = vx' \Rightarrow \frac{dy'}{dx'} = v + x'\frac{dv}{dx'}$

(22)

$$\Rightarrow v + x'\frac{dv}{dx'} = -\frac{7x'-3vx'}{3x'-7vx'} = -\left(\frac{7-3v}{3-7v}\right)$$

$$\Rightarrow x'\frac{dv}{dx'} = -\frac{7-3v}{3-7v} - v = -\left[\frac{7-\cancel{3v}+\cancel{3v}-7v^2}{3-7v}\right] = -\left[\frac{7-7v^2}{3-7v}\right]$$

$$\Rightarrow x'\frac{dv}{dx'} = \frac{7(v^2-1)}{3-7v}$$

$$\Rightarrow \frac{3-7v}{(v^2-1)}\,dv = 7\frac{dx}{x'}$$

Integrating both sides.

$$\int\left[\frac{-2}{v-1} - \frac{5}{v+1}\right]dv = 7\int\frac{1}{x'}\,dx' + c$$

$$\frac{3-7v}{v^2-1} = \frac{A}{v-1} + \frac{B}{v+1}$$

$$3-7v = A(v+1) + B(v-1)$$

$$\Rightarrow \boxed{A=-2} \quad \boxed{B=-5}$$

$$\Rightarrow -2\log(v-1) - 5\log(v+1) = 7\log(x') + \log c$$

$$\Rightarrow \log c = 2\log(v-1) + 5\log(v+1) + 7\log(x')$$

$$= \log(v-1)^2 + \log(v+1)^5 + \log(x')^7$$

$$\Rightarrow \log\left[(v-1)^2(v+1)^5(x')^7\right] = \log c$$

$$\Rightarrow (v-1)^2(v+1)^5(x')^7 = c$$

$$\Rightarrow \left(\frac{y'}{x'}-1\right)^2\left(\frac{y'}{x'}+1\right)^5(x')^7 = c$$

$$\Rightarrow \frac{(y'-x')^2(y'+x')^5\cancel{(x')^7}}{\cancel{(x')^7}} = c$$

$$\Rightarrow (y'-x')^2(y'+x')^5 = c$$

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$\Rightarrow \frac{dy'}{dx'}$ Now, we have $x = x'+h = x'+1$ $\left[\because h=1, k=0\right]$

$y = y'+k = y'+0$

$\Rightarrow y'-x' = (y-x+1)$

& $y'+x' = (x+y-1)$

$\therefore (y'-x')^2(y'+x')^5 = c$

$\Rightarrow (y-x+1)^2(y+x-1)^5 = c$

which is the required solⁿ

Ex. 2 solve $(2x+4y+3)\frac{dy}{dx} = x+2y+1$

solⁿ:- Given eqⁿ is

$$\frac{dy}{dx} = \frac{x+2y+1}{2x+4y+3} = \frac{x+2y+1}{2(x+2y)+3} \quad \text{—— (1)}$$

Now $\frac{a}{a'} = \frac{1}{2}$, $\frac{b}{b'} = \frac{1}{2}$

so that put $x+2y = t \Rightarrow 1+2\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{dt}{dx}-1\right)$$

$$(1) \Rightarrow \frac{1}{2}\left(\frac{dt}{dx}-1\right) = \frac{t+1}{2t+3} = \frac{2t+2}{2t+3}$$

$$\Rightarrow \frac{dt}{dx} = \frac{2t+1}{2t+3}+1 = \frac{2t+2+2t+3}{2t+3} = \frac{4t+5}{2t+3}$$

$$\Rightarrow \frac{dt}{dx} = \frac{4t+5}{2t+3} \Rightarrow \left(\frac{2t+3}{4t+5}\right)dt = dx$$

$$\Rightarrow \frac{1}{2}\int\left(\frac{4t+5+1}{4t+5}\right)dt = \int dx + c$$

$$\Rightarrow \frac{1}{2}\int\left[\frac{4t+5}{4t+5}+\frac{1}{4t+5}\right]dt = x+c$$

$$\Rightarrow \frac{1}{2}\int\left[1+\frac{1}{4t+5}\right]dt = x+c$$

$$\Rightarrow \frac{1}{2}\left[t+\log(4t+5)\left(\frac{1}{4}\right)\right] = x+c$$

$$\Rightarrow \frac{t}{2}+\frac{1}{8}\log(4t+5) = x+c$$

$$\Rightarrow 4t+\log(4t+5) = 8x+c' \quad ; \quad 8c = c'$$

$$\Rightarrow 4(x+2y)+\log(4x+8y+5) = 8x+c'$$

$$\Rightarrow 8y-4x+\log(4x+8y+5) = c$$

which is the required sol^n.

Ex. 3 solve $(x+2y)(dx-dy) = dx+dy$.

sol^n:- Given differential eq^n is

$$(x+2y-1)dx = (x+2y+1)dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+2y-1}{x+2y+1} \quad —— (1)$$

put $x+2y = t \Rightarrow 1+2\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{dt}{dx}-1\right)$$

$$\because \frac{a}{a'} = \frac{1}{1} = \frac{b}{b'}$$
$$\Rightarrow \frac{a}{a'} = \frac{b}{b'} = 1$$

$$(1) \Rightarrow \frac{1}{2}\left(\frac{dt}{dx}-1\right) = \frac{t-1}{t+1} = \frac{2(t-1)}{t+1}$$

$$\Rightarrow \frac{dt}{dx}-1 = \frac{2t-2}{t+1}$$